Stringy Differential Geometry and Double Field Theory

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Prologue

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- In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.
  - Diffeomorphism: $\partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$
  - $\nabla_\lambda g_{\mu\nu} = 0,\ \Gamma^\lambda_{[\mu\nu]} = 0 \longrightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$
  - Curvature: $[\nabla_\mu, \nabla_\nu] \longrightarrow R_{\kappa\lambda\mu\nu} \longrightarrow R$
- On the other hand, string theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ on an equal footing, as they – so called NS-NS sector – form a multiplet of T-duality.
- This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.
- Basically, Riemannian geometry is for *Particle* theory. *String* theory requires a novel differential geometry which geometrizes the whole NS-NS sector.

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- My talk today aims to introduce such a **Stringy Geometry** which is defined in **doubled-yet-gauged** spacetime.

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- Gauge symmetry is a 'non-physical' symmetry.
- It is a redundant symmetry of Lagrangian, not a physical symmetry of Nature.
- All the physical quantities are gauge invariant. Gauge transformations do not change any physics.
- However, ironically and intriguigly enough, Gauge Symmetry has been a key principle in modern physics and has lead to the success of the Standard Model.
- In particular, in four-dimensional spacetime photon has 'two' physical degrees of freedom, but can be best described by a 'four' component vector.
- One of the main messages of this talk:

  D-dimensional spacetime may be better understood in terms of **doubled-yet-gauged** $(D + D)$ number of coordinates, at least for String Theory.

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Talk based on works with Imtak Jeon & Kanghoon Lee

- Differential geometry with a projection: Application to double field theory
  arXiv:1011.1324 JHEP
- Double field formulation of Yang-Mills theory arXiv:1102.0419 PLB
- Stringy differential geometry, beyond Riemann arXiv:1105.6294 PRD
- Incorporation of fermions into double field theory arXiv:1109.2035 JHEP
- Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity
  arXiv:1112.0069 PRD Rapid Comm.
- Ramond-Ramond Cohomology and O(D,D) T-duality arXiv:1206.3478 JHEP
- **Stringy Unification of Type IIA and IIB Supergravities under $\mathcal{N} = 2$ $D = 10$ Supersymmetric Double Field Theory** arXiv:1210.5078 PLB
- Comments on double field theory and diffeomorphisms arXiv:1304.5946 JHEP
- Covariant action for a string in doubled yet gauged spacetime arXiv:1307.8377 NPB

Parallel works on U-duality

- U-geometry: SL(5) with Yoonji Suh arXiv:1302.1652 JHEP
- M-theory and F-theory from a Duality Manifest Action
  with Chris Blair and Emanuel Malek arXiv:1311.5109 JHEP
- U-gravity: SL(N) with Yoonji Suh arXiv:1402.5027 JHEP

Double Field Theory by Hull & Zwiebach (Hohm), *c.f.* Siegel

- With a "generalized metric" Duff and a redefined dilaton:

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \qquad e^{-2d} = \sqrt{-g}e^{-2\phi}$$

- DFT Lagrangian constructed by Hull & Zwiebach (Hohm) reads

$$L_{\mathrm{DFT}} = e^{-2d}\left[\mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right) + 4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\right]$$

- Spacetime is formally doubled, $y^A = (\tilde{x}_\mu, x^\nu)$, $A = 1, 2, \cdots, D{+}D$.
- T-duality is manifestly realized as usual $\mathbf{O}(D, D)$ rotations Tseytlin, Siegel

$$\mathcal{H}_{AB} \longrightarrow M_A{}^C M_B{}^D \mathcal{H}_{CD}\,, \qquad d \longrightarrow d\,, \qquad M \in \mathbf{O}(D, D)\,.$$

- Yet, DFT (for NS-NS sector) is a D-dimensional theory written in terms of $(D + D)$-dimensional language, i.e. tensors.
- All the fields must live on a D-dimensional null hyperplane or 'section', subject to

$$\partial_A\partial^A = 2\frac{\partial^2}{\partial x^\mu \partial\tilde{x}_\mu} \equiv 0 \quad : \quad \text{section condition}$$

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- Up to $\mathbf{O}(D,D)$ rotation, we may fix the section, or choose to set

$$\frac{\partial}{\partial\tilde{x}_\mu} \equiv 0\,.$$

- Then DFT reduces to the well-known effective action within 'Riemannian' setup:

$$L_{\rm DFT} \Longrightarrow L_{\rm eff.} = \sqrt{-g}e^{-2\phi}\left(R_g + 4(\partial\phi)^2 - \tfrac{1}{12}H^2\right).$$

where the diffeomorphism and the B-field gauge symmetry are 'tamed' under our control,

$$x^\mu \to x^\mu + \delta x^\mu\,, \qquad B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu\,.$$

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$$x^\mu \rightarrow x^\mu + \delta x^\mu\,, \qquad B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu\,.$$

Double Field Theory by Hull & Zwiebach (Hohm), *c.f.* Siegel

- With a "generalized metric" Duff and a redefined dilaton:

$$\mathcal{H}_{AB} = \left(\begin{array}{cc} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{array}\right), \qquad e^{-2d} = \sqrt{-g}e^{-2\phi}$$

- DFT Lagrangian constructed by Hull & Zwiebach (Hohm) reads

$$L_{\mathrm{DFT}} = e^{-2d}\left[\mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right) + 4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\right]$$

- On the other hand, in the above formulation of DFT, the diffeomorphism and the B-field gauge symmetry are rather unclear, while $\mathbf{O}(D,D)$ T-duality is manifest.
- The above expression may be analogous to the case of writing the Riemannian scalar curvature, R, in terms of the metric and its derivative.
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  - **Semi-covariant curvature**
  - **And their complete covariantization via 'projection'**

c.f. Alternative approaches: Berman-Blair-Malek-Perry, Cederwall, Geissbuhler, Marques et al.

Question: Is DFT a mere reformulation of SUGRA?

- YES, if we take the following as a definition of the generalized metric,

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} .$$

- NO, if we define the generalized metric as a symmetric $\mathbf{O}(D, D)$ element,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA} , \qquad \mathcal{H}_A{}^C \mathcal{H}_B{}^D \mathcal{J}_{CD} = \mathcal{J}_{AB} ,$$

  where $\mathcal{J}$ denotes the $\mathbf{O}(D, D)$ invariant constant metric.

- With this abstract definition, DFT as well as a worldsheet sigma model (which I will discuss later) perfectly make sense.

- It may then describe a novel non-Riemannian string theory backgrounds, e.g.

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Geometric Constitution of Double Field Theory

- *Notation*

  Capital Latin alphabet letters denote the $\mathbf{O}(D, D)$ vector indices, i.e. $A, B, C, \cdots = 1, 2, \cdots, D{+}D$, which can be freely raised or lowered by the $\mathbf{O}(D, D)$ invariant constant metric,

$$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

- *Doubled-yet-gauged spacetime*

The spacetime is formally doubled, being $(D{+}D)$-dimensional.

However, **the doubled spacetime is gauged** : the coordinate space is equipped with an *equivalence relation*,

$$x^{A} \;\sim\; x^{A} + \phi\partial^{A}\varphi\,,$$

which we call '*coordinate gauge symmetry*'.

Note that ϕ and φ are arbitrary functions in DFT.

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which we call '*coordinate gauge symmetry*'.

Note that ϕ and φ are arbitrary functions in DFT.

Each equivalence class, or gauge orbit, represents a single physical point.

Diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the gauge orbits.

- *Realization of the coordinate gauge symmetry.*

The equivalence relation is realized in DFT by enforcing that, arbitrary functions and their arbitrary derivatives, denoted here collectively by Φ, are invariant under the coordinate gauge symmetry shift,

$$\Phi(x+\Delta)=\Phi(x)\,,\qquad\qquad \Delta^{A}=\phi\partial^{A}\varphi\,.$$

- *Section condition.*

The invariance under the coordinate gauge symmetry can be shown to be equivalent to the **section condition** ,

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Explicitly, acting on arbitrary functions, Φ, Φ', and their products, we have

$$\partial_A \partial^A \Phi = 0 \qquad \text{(weak constraint)}\,,$$

$$\partial_A \Phi \partial^A \Phi' = 0 \qquad \text{(strong constraint)}\,.$$

- *Diffeomorphism.*

Diffeomorphism symmetry in $\mathbf{O}(D,D)$ DFT is generated by a generalized Lie derivative Siegel, Courant, Grana

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n} := X^B\partial_B T_{A_1\cdots A_n} + \omega_T\,\partial_B X^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}X_B - \partial_B X_{A_i})T_{A_1\cdots A_{i-1}}{}^B{}_{A_{i+1}\cdots A_n}\,,$$

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where ω_T denotes the weight.

In particular, the generalized Lie derivative of the $\mathbf{O}(D,D)$ invariant metric is trivial,

$$\hat{\mathcal{L}}_X \mathcal{J}_{AB} = 0\,.$$

The commutator is closed by C-bracket Hull-Zwiebach

$$\left[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y\right] = \hat{\mathcal{L}}_{[X,Y]_{\mathrm{C}}}\,, \qquad [X,Y]^A_{\mathrm{C}} = X^B\partial_B Y^A - Y^B\partial_B X^A + \tfrac{1}{2}Y^B\partial^A X_B - \tfrac{1}{2}X^B\partial^A Y_B\,.$$

- *Dilaton and a pair of two-index projectors.*

The geometric objects in DFT consist of a dilation, d, and a pair of symmetric projection operators,

$$P_{AB} = P_{BA}\,, \qquad \bar{P}_{AB} = \bar{P}_{BA}\,, \qquad P_A{}^B P_B{}^C = P_A{}^C\,, \qquad \bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C\,.$$

Further, the projectors are orthogonal and complementary,

$$P_A{}^B \bar{P}_B{}^C = 0\,, \qquad P_{AB} + \bar{P}_{AB} = \mathcal{J}_{AB}\,.$$

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Remark: The difference of the two projectors, $P_{AB} - \bar{P}_{AB} = \mathcal{H}_{AB}$, corresponds to the "generalized metric" which can be also independently defined as a symmetric $\mathbf{O}(D, D)$ element, i.e. $\mathcal{H}_{AB} = \mathcal{H}_{BA}$, $\mathcal{H}_A{}^B \mathcal{H}_B{}^C = \delta_A{}^C$. However, in supersymmetric double field theories it appears that the projectors are more fundamental than the "generalized metric".

- *Integral measure.*

While the projectors are weightless, the dilation gives rise to the $\mathbf{O}(D,D)$ invariant integral measure with weight one, after exponentiation,

$$e^{-2d}\,.$$

- *Semi-covariant derivative and semi-covariant Riemann curvature.*

We define a semi-covariant derivative,

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega_T \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,,$$

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and a semi-covariant Riemann curvature,

$$S_{ABCD} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^E{}_{AB}\Gamma_{ECD}\right) .$$

Here R_{ABCD} denotes the ordinary "field strength" of a connection,

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^E \Gamma_{BED} - \Gamma_{BC}{}^E \Gamma_{AED} .$$

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As I will explain shortly, we may determine the (torsionelss) connection:

$$\begin{aligned}\Gamma_{CAB} = \quad & 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} \\ & -\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) \,,\end{aligned}$$

which is the DFT generalization of the Christoffel connection.

The semi-covariant derivative then obeys the Leibniz rule and annihilates the $\mathbf{O}(D,D)$ invariant constant metric,

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A crucial defining property of the semi-covariant "Riemann" curvature is that, under arbitrary transformation of the connection, it transforms as total derivative,

$$\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB}\,.$$

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Further, the semi-covariant "Riemann" curvature satisfies precisely the same symmetric properties as the ordinary Riemann curvature,

$$S_{ABCD} = S_{[AB][CD]} = S_{CDAB}\,, \qquad S_{[ABC]D} = 0\,,$$

as well as additional identities concerning the projectors,

$$P_I{}^A P_J{}^B \bar{P}_K{}^C \bar{P}_L{}^D S_{ABCD}{=}0\,, \qquad P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} = 0\,.$$

It follows that

$$S^{AB}{}_{AB}{=}0\,.$$

- *The uniqueness of the torsionless connection.*

The connection is the unique solution to the following five constraints:

$$\nabla_A P_{BC} = 0 \,, \qquad \nabla_A \bar{P}_{BC} = 0 \,,$$

$$\nabla_A d = -\tfrac{1}{2} e^{2d} \nabla_A (e^{-2d}) = \partial_A d + \tfrac{1}{2} \Gamma^B{}_{BA} = 0 \,,$$

$$\Gamma_{ABC} + \Gamma_{ACB} = 0 \,,$$

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- The first two relations are the compatibility conditions with all the geometric objects , or NS-NS sector, in DFT.

- The third constraint is the compatibility condition with the $\mathbf{O}(D, D)$ invariant constant metric, *i.e.* $\nabla_A \mathcal{J}_{BC} = 0$.

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- The next cyclic property makes the semi-covariant derivative compatible with the generalized Lie derivative as well as with the C-bracket,

$$\hat{\mathcal{L}}_X(\partial) = \hat{\mathcal{L}}_X(\nabla)\,, \qquad [X,Y]_{\mathrm{C}}(\partial) = [X,Y]_{\mathrm{C}}(\nabla)\,.$$

- The last formulae are projection conditions which we impose intentionally in order to ensure the uniqueness.

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- *Six-index projection operators.*

The six-index projection operators are explicitly,

$$\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D} \,,$$

$$\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D} \,,$$

which satisfy the 'projection' properties,

$$\mathcal{P}_{ABC}{}^{DEF} \mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{ABC}{}^{GHI} \,, \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF} \bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{ABC}{}^{GHI} \,.$$

Further, they are symmetric and traceless,

$$\mathcal{P}_{ABCDEF} = \mathcal{P}_{DEFABC} \,, \qquad \mathcal{P}_{ABCDEF} = \mathcal{P}_{A[BC]D[EF]} \,, \qquad P^{AB} \mathcal{P}_{ABCDEF} = 0 \,,$$

$$\bar{\mathcal{P}}_{ABCDEF} = \bar{\mathcal{P}}_{DEFABC} \,, \qquad \bar{\mathcal{P}}_{ABCDEF} = \bar{\mathcal{P}}_{A[BC]D[EF]} \,, \qquad \bar{P}^{AB} \bar{\mathcal{P}}_{ABCDEF} = 0 \,.$$

Crucially, the projection operator dictates the anomalous terms in the diffeomorphic transformations of the semi-covariant derivative and the semi-covariant Riemann curvature,

$$(\delta_X - \hat{\mathcal{L}}_X)\nabla_C T_{A_1\cdots A_n} = \sum_{i=1}^{n} 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BDEF}\partial_D\partial_E X_F\, T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,,$$

$$(\delta_X - \hat{\mathcal{L}}_X)S_{ABCD} = 2\nabla_{[A}\Big((\mathcal{P}+\bar{\mathcal{P}})_{B][CD]}{}^{EFG}\partial_E\partial_F X_G\Big) + 2\nabla_{[C}\Big((\mathcal{P}+\bar{\mathcal{P}})_{D][AB]}{}^{EFG}\partial_E\partial_F X_G\Big).$$

- *Complete covariantizations.*

Both the semi-covariant derivative and the semi-covariant Riemann curvature can be fully covariantized, through appropriate contractions with the projectors:

$$P_{C}{}^{D}\bar{P}_{A_1}{}^{B_1}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_{D}T_{B_1\cdots B_n}\,,\qquad \bar{P}_{C}{}^{D}P_{A_1}{}^{B_1}\cdots P_{A_n}{}^{B_n}\nabla_{D}T_{B_1\cdots B_n}\,,$$

$$P^{AB}\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}T_{BD_1\cdots D_n}\,,\qquad \bar{P}^{AB}P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}\nabla_{A}T_{BD_1\cdots D_n}\qquad(\text{divergences})\,,$$

$$P^{AB}\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}\nabla_{B}T_{D_1\cdots D_n}\,,\qquad \bar{P}^{AB}P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}\nabla_{A}\nabla_{B}T_{D_1\cdots D_n}\qquad(\text{Laplacians})\,,$$

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and

$$P_A{}^C\bar{P}_B{}^D S_{CED}{}^E\qquad\text{(“Ricci” curvature)}\,,$$

$$(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\qquad\text{(scalar curvature)}\,.$$

- *Action.*

The action of $\mathbf{O}(D, D)$ DFT is given by the fully covariant scalar curvature,

$$\int_{\Sigma_D} e^{-2d}(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\,,$$

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Note: It is precisely the above expression that allows the ‘1.5 formalism’ to work in the full order supersymmetric extensions of $\mathcal{N} = 1, 2,\ D = 10$ Jeon-Lee-JHP

- *Section.*

Up to $\mathbf{O}(D,D)$ duality rotations, the solution to the section condition is unique. It is a D-dimensional section, Σ_D, characterized by the independence of the dual coordinates, i.e.

$$\frac{\partial}{\partial\tilde{x}_\mu}\equiv 0\,,$$

while the whole doubled coordinates are given by

$$x^A=(\tilde{x}_\mu,x^\nu)\,,$$

where μ,ν are now D-dimensional indices.

- *Riemannian reduction.*

  To perform the Riemannian reduction to the D-dimensional section, Σ_D, we parametrize the dilation and the projectors in terms of D-dimensional Riemannian metric, $g_{\mu\nu}$, ordinary dilaton, ϕ, and a Kalb-Ramond two-form potential, $B_{\mu\nu}$,

$$P_{AB} - \bar{P}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \qquad e^{-2d} = \sqrt{|g|}e^{-2\phi}\,.$$

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  The DFT scalar curvature then reduces upon the section to

$$(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\Big|_{\Sigma_D}=R_g+4\Delta\phi-4\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\,,$$

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  Up to field redefinitions, the above is the most general parametrization of the “generalized metric", $\mathcal{H}_{AB}=P_{AB}-\bar{P}_{AB}$, when its upper left $D\times D$ block is non-degenerate.

- *Non-Riemannian backgrounds.*

When the upper left $D \times D$ block of $\mathcal{H}_{AB} = (P - \bar{P})_{AB}$ is degenerate – where g^{-1} might be positioned – the Riemannian metric ceases to exist upon the section, Σ_D.

Nevertheless, DFT and a doubled sigma model –which I will discuss later– have no problem with describing such a non-Riemannian background.

An extreme example of such a non-Riemannian background is the flat background where

$$\mathcal{H}_{AB} = (P - \bar{P})_{AB} = \mathcal{J}_{AB}\,.$$

This is a vacuum solution to the bosonic DFT and the corresponding doubled sigma model reduces to a certain 'chiral' sigma model.

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Allowing non-Riemannian backgrounds, DFT is NOT a mere reformulation of SUGRA. It describes a new class of string theory backgrounds. *c.f.* Gomis-Ooguri

Supersymmetric Extension

Based on the differential geometry I just described,

after incorporating fermions and R-R sector,

it is possible to construct, to the full order in fermions,

Type II, or $\mathcal{N}=2$, $D=10$ Supersymmetric Double Field Theory

of which the Lagrangian reads

$$\mathcal{L}_{\mathrm{Type\,II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD}+\tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})-i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho-i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}}-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho'+i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]$$

Jeon-Lee-Suh-JHP

Symmetries of $\mathcal{N}=2$ $D=10$ SDFT

- **O**(D,D) **T-duality**
- **Gauge symmetries**

  1. **DFT-diffeomorphism (generalized Lie derivative)**
  2. **A *pair* of local Lorentz symmetries, Spin**$(1,D{-}1)_L \times$ **Spin**$(D{-}1,1)_R$
  3. **local** $\mathcal{N}=2$ **SUSY with** 32 **supercharges.**

- **All the bosonic symmetries are realized manifestly and simultaneously.**
- **For this, it is crucial to have the right field variables:**

$$d\,,\quad V_{Ap}\,,\quad \bar{V}_{A\bar{p}}\,,\quad \mathcal{C}^{\alpha}{}_{\bar{\alpha}}\,,\quad \rho^{\alpha}\,,\quad \rho'^{\bar{\alpha}}\,,\quad \psi^{\alpha}_{\bar{p}}\,,\quad \psi'^{\bar{\alpha}}_{p}$$

which are O(D,D) **covariant genuine DFT-field-variables, and *a priori* they are NOT Riemannian, such as metric,** B**-field, R-R** p**-forms.**

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For details of the supersymmetric construction,

I refer the audience to the talk by Dr. Imtak Jeon

on Friday, PA3-2 18:30-21:30

Stringy Unification of Type IIA and IIB Supergravities under
$\mathcal{N}=2$ $D=10$ *Supersymmetric Double Field Theory*

Worldsheet Perspective

String propagates in doubled-yet-gauged spacetime

- The section condition is equivalent to the 'coordinate gauge symmetry', 1304.5946

$$x^M \sim x^M + \varphi\partial^M\varphi' .$$

A 'physical point' is one-to-one identified with a 'gauge orbit' in coordinate space.

- The coordinate gauge symmetry can be concretely realized on worldsheet, 1307.8377

$$\mathcal{S} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}\,, \qquad \mathcal{L} = -\tfrac{1}{2}\sqrt{-h}h^{ij}D_iX^M D_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

where

$$D_iX^M = \partial_iX^M - \mathcal{A}_i^M\,, \qquad \mathcal{A}_i^M\partial_M \equiv 0\,.$$

- The Lagrangian is quite symmetric thanks to the auxiliary gauge field, $\mathcal{A}_i^M$:
  - String worldsheet diffeomorphisms plus Weyl symmetry (as usual)
  - $\mathbf{O}(D,D)$ T-duality
  - Target spacetime diffeomorphisms
  - The coordinate gauge symmetry

c.f. Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh

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- For example, under target spacetime 'finite' diffeomorphism *à la* Zwiebach-Hohm

$$L_M{}^N := \partial_A X'^B\,,\qquad\qquad \bar{L} := \mathcal{J} L^t \mathcal{J}^{-1}\,,$$

$$F := \tfrac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right)\,,\qquad\qquad \bar{F} := \mathcal{J} F^t \mathcal{J}^{-1} = \tfrac{1}{2}\left(L^{-1}\bar{L} + \bar{L}L^{-1}\right) = F^{-1}\,,$$

each field transforms as

$$\begin{array}{lcl}
X^M & \longrightarrow & X'^M(X)\,, \\
\mathcal{H}_{MN}(X) & \longrightarrow & \mathcal{H}'_{MN}(X') = \bar{F}_M{}^K \bar{F}_N{}^L \mathcal{H}_{KL}(X)\,, \\
\mathcal{A}^M & \longrightarrow & \mathcal{A}'^M = \mathcal{A}^N F_N{}^M + \mathrm{d}X^N (L-F)_N{}^M \qquad : \quad \mathcal{A}'^M \partial'_M \equiv 0\,, \\
DX^M & \longrightarrow & D'X'^M = DX^N F_N{}^M\,,
\end{array}$$

such that the worldsheet action remains invariant, up to total derivatives.

String propagates in doubled-yet-gauged spacetime

- The Equation Of Motion for X^L can be conveniently organized in terms of our DFT-Christoffel connection:

$$\frac{1}{\sqrt{-h}}\partial_i\left(\sqrt{-h}D^iX^M\mathcal{H}_{ML}+\epsilon^{ij}\partial_i\mathcal{A}_{jL}\right)-2\Gamma_{LMN}(PD_iX)^M(\bar{P}D^iX)^N=0\,,$$

which is comparable to the *geodesic motion* of a point particle, $\ddot{Y}^\lambda+\Gamma^\lambda_{\mu\nu}\dot{Y}^\mu\dot{Y}^\nu=0.$

- The EOM of $\mathcal{A}_i^M$ implies *a priori*,

$$\delta\mathcal{A}_{iM}\left(\mathcal{H}^M{}_ND^iX^N+\frac{1}{\sqrt{-h}}\epsilon^{ij}D_jX^M\right)=0\,.$$

Especially, for the case of the 'non-degenerate' Riemannian background, a complete self-duality follows

$$\mathcal{H}^M{}_ND^iX^N+\frac{1}{\sqrt{-h}}\epsilon^{ij}D_jX^M=0\,.$$

- Finally, the EOM of h_{ij} gives the Virasoro constraints,

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- After parametrization, $X^M = (\tilde{Y}_\mu, Y^\nu)$, $\mathcal{H}_{MN}(G,B)$, and integrating out $\mathcal{A}_i^M$, it can produce either the standard string action for the 'non-degenerate' Riemannian case,

$$\frac{1}{4\pi\alpha'}\mathcal{L} \equiv \frac{1}{2\pi\alpha'}\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_i Y^\mu \partial_j Y^\nu G_{\mu\nu}(Y) + \tfrac{1}{2}\epsilon^{ij}\partial_i Y^\mu \partial_j Y^\nu B_{\mu\nu}(Y) + \tfrac{1}{2}\epsilon^{ij}\partial_i \tilde{Y}_\mu \partial_j Y^\mu\right] ,$$

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or chiral actions for 'degenerate' non-Riemannian cases, e.g. for $\mathcal{H}_{AB} = \mathcal{J}_{AB}$,

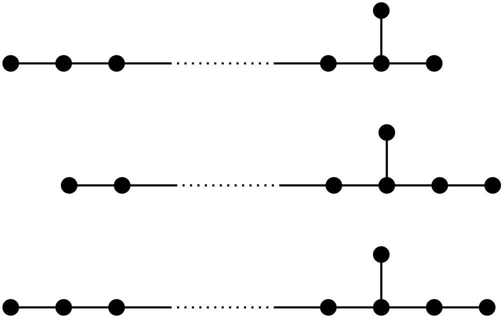
$$\frac{1}{4\pi\alpha'}\mathcal{L} \equiv \frac{1}{4\pi\alpha'}\epsilon^{ij}\partial_i \tilde{Y}_\mu \partial_j Y^\mu\,, \qquad\qquad \partial_i Y^\mu + \frac{1}{\sqrt{-h}}\epsilon_i{}^j\partial_j Y^\mu = 0\,.$$

c.f. Gomis-Ooguri

U-duality

Parallel to the stringy differential geometry for O(D, D) T-duality,

it is possible to construct M-theoretic differential geometry for each U-duality group.



	$A_{N-1} \equiv \mathbf{sl}(N)$
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Table: Dynkin diagrams for A_{N-1}, D_{N-1}, E_{N-1} and E_N

- E_{11} : conjectured to be the ultimate duality group. West
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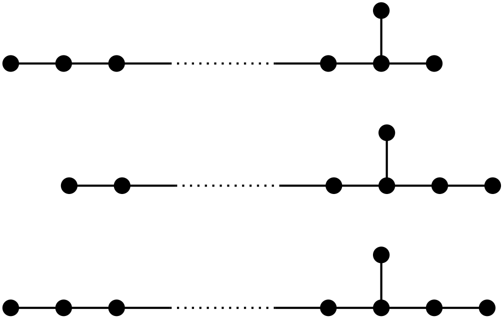

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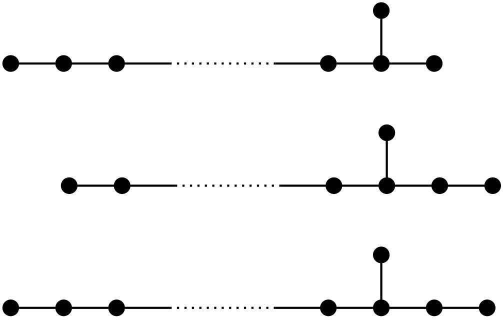

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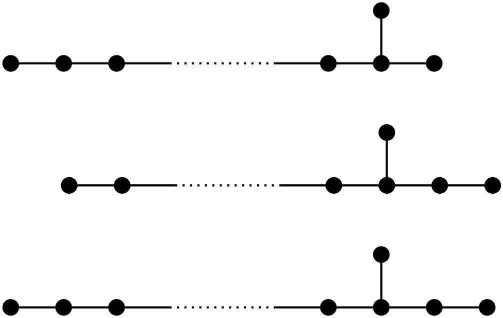

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  - Extended-yet-gauged spacetime (*c.f.* Berman-Perry for $N=5$),

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